Failure Theories – Static Loads

Static load – a stationary load that is gradually applied having an unchanging magnitude and direction

Failure – A part is permanently distorted and will not function properly.

A part has been separated into two or more pieces.

Material Strength

S_y = Yield strength in tension, $S_{yt} = S_{yc}$

S_{ys} = Yield strength in shear

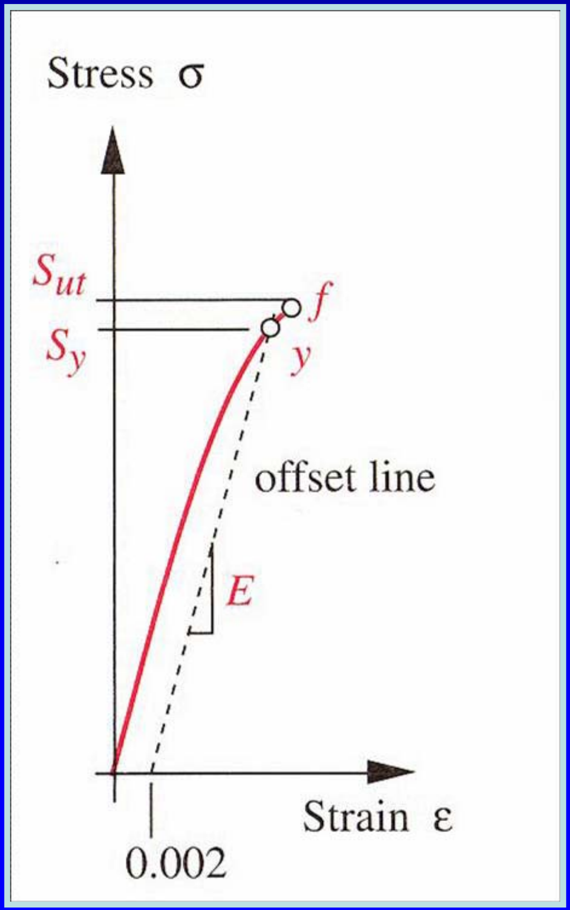
S_u = Ultimate strength in tension, S_{ut}

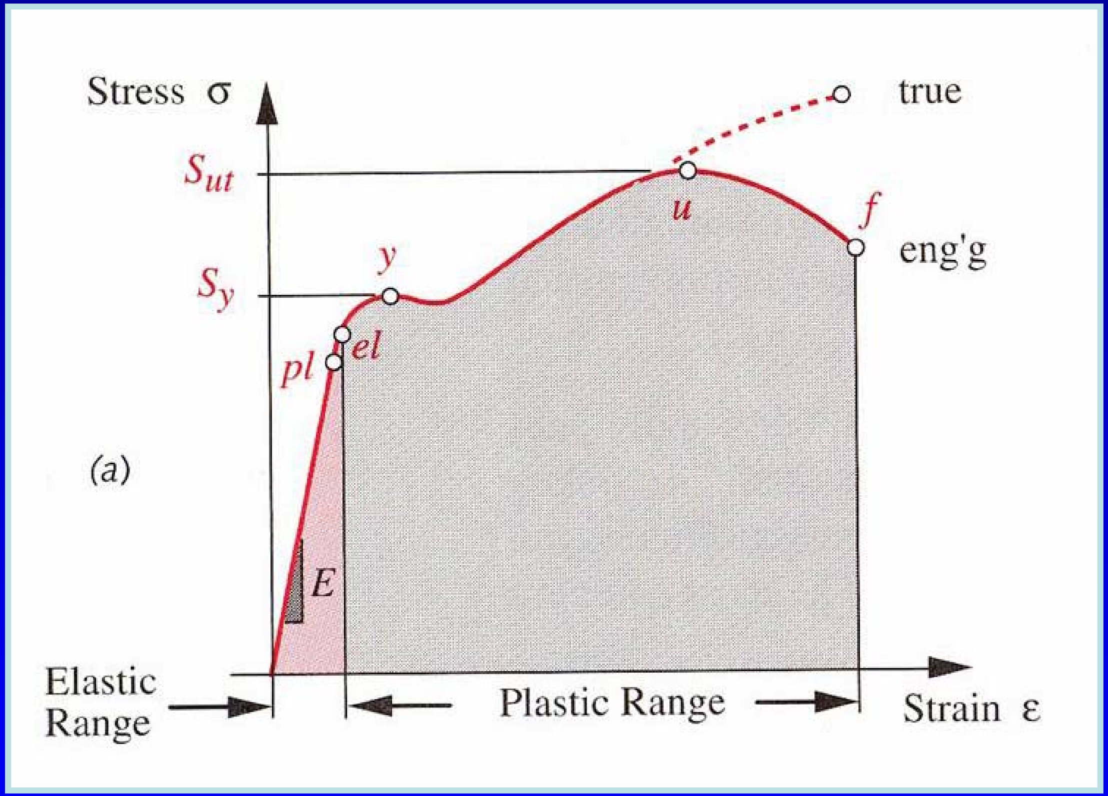
S_{uc} = Ultimate strength in compression

S_{us} = Ultimate strength in shear = .67 S_u

Ductile and Brittle Materials

A ductile material deforms significantly before fracturing. Ductility is measured by % elongation at the fracture point. Materials with 5% or more elongation are considered ductile.





Brittle material yields very little before fracturing, the yield strength is approximately the same as the ultimate strength in tension. The ultimate strength in compression is much larger than the ultimate strength in tension.

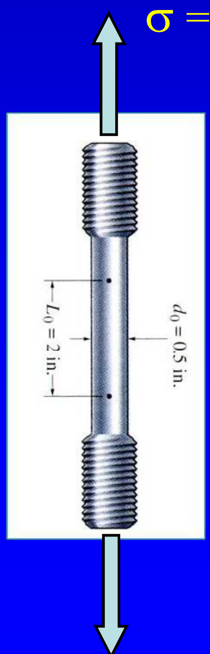
Failure Theories – Ductile Materials

Yield strength of a material is used to design components made of ductile material

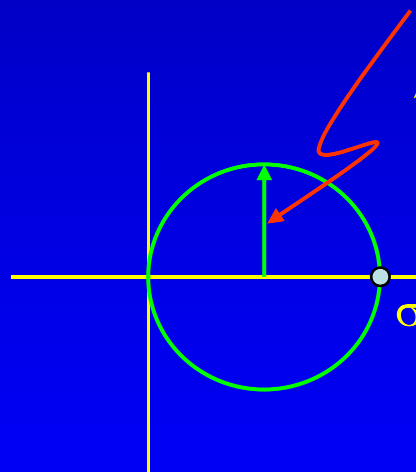
- Maximum shear stress theory (Tresca 1886)

$$(\tau_{max})_{component} > (\tau)_{\text{obtained from a tension test at the yield point}} \longrightarrow \text{Failure}$$

$\sigma = S_y$



$\sigma = S_y$



To avoid failure

$$(\tau_{max})_{component} < \frac{S_y}{2}$$

$$\tau_{max} = \frac{S_y}{2\,n}$$

Design equation

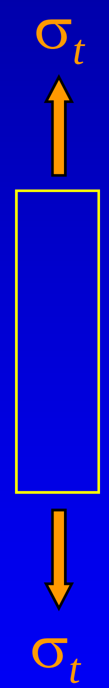
n = Safety factor

Failure Theories – Ductile Materials

- Distortion energy theory (von Mises-Hencky)

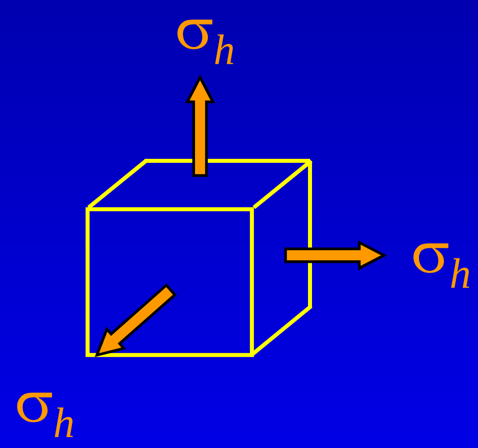
Simple tension test $\rightarrow$ $(S_y)_t$

Hydrostatic state of stress $\rightarrow$ $(S_y)_h$



$$(S_y)_h >> (S_y)_t$$

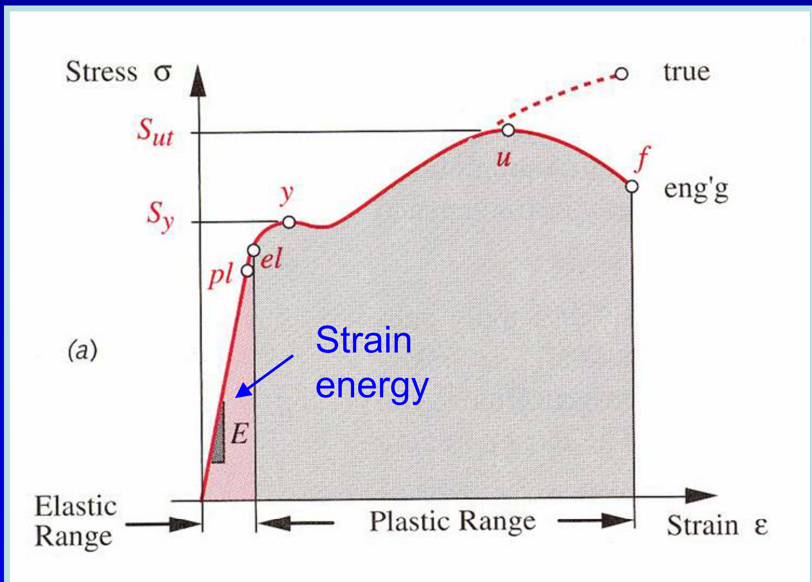
Distortion contributes to failure much more than change in volume.



(total strain energy) – (strain energy due to hydrostatic stress) = strain energy due to angular distortion **>** strain energy obtained from a tension test at the yield point $\rightarrow$ ***failure***

Failure Theories – Ductile Materials

The area under the curve in the elastic region is called the Elastic Strain Energy.



$$U = \tfrac{1}{2}\,\sigma\varepsilon$$

3D case

$$U_T = \tfrac{1}{2}\,\sigma_1\varepsilon_1 + \tfrac{1}{2}\,\sigma_2\varepsilon_2 + \tfrac{1}{2}\,\sigma_3\varepsilon_3$$

Stress-strain relationship

$$\varepsilon_1 = \frac{\sigma_1}{E} - \nu\frac{\sigma_2}{E} - \nu\frac{\sigma_3}{E}$$

$$\varepsilon_2 = \frac{\sigma_2}{E} - \nu\frac{\sigma_1}{E} - \nu\frac{\sigma_3}{E}$$

$$\varepsilon_3 = \frac{\sigma_3}{E} - \nu\frac{\sigma_1}{E} - \nu\frac{\sigma_2}{E}$$

$$U_T = \frac{1}{2E}\left[(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - 2\nu\,(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3) \right]$$

Failure Theories – Ductile Materials

Distortion strain energy = total strain energy – hydrostatic strain energy

$$U_d = U_T - U_h$$

$$U_T = \frac{1}{2E}\left[(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - 2v\,(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3)\right] \qquad (1)$$

Substitute $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_h$

$$U_h = \frac{1}{2E}\left[(\sigma_h^2 + \sigma_h^2 + \sigma_h^2) - 2v\,(\sigma_h\sigma_h + \sigma_h\sigma_h + \sigma_h\sigma_h)\right]$$

Simplify and substitute $\sigma_1 + \sigma_2 + \sigma_3 = 3\sigma_h$ into the above equation

$$U_h = \frac{3\sigma_h^2}{2E}(1 - 2v) = \frac{(\sigma_1 + \sigma_2 + \sigma_3)^2 (1 - 2v)}{6E}$$

Subtract the hydrostatic strain energy from the total energy to obtain the distortion energy

$$U_d = U_T - U_h = \frac{1+v}{6E}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2 \right] \qquad (2)$$

Failure Theories – Ductile Materials

Strain energy from a tension test at the yield point

$\sigma_1 = S_y$ and $\sigma_2 = \sigma_3 = 0$ Substitute in equation (2)

$$U_d = U_T - U_h = \frac{1+\nu}{6E}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2\right] \quad (2)$$

$$U_{test} = (S_y)^2 \frac{1+\nu}{3E}$$

To avoid failure, $U_d < U_{test}$

$$\left[\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2}{2}\right]^{1/2} < S_y$$

Failure Theories – Ductile Materials

$$\left[\frac{(\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2}{2}\right]^{1/2} < S_y$$

2D case, $\sigma_3 = 0$

$$(\sigma_1^{\,2} - \sigma_1\sigma_2 + \sigma_2^{\,2})^{1/2} < S_y = \sigma'$$

Where σ' is von Mises stress

$$\sigma' = \frac{S_y}{n}$$

Design equation

Failure Theories – Ductile Materials

Pure torsion, $\tau = \sigma_1 = -\sigma_2$

$$(\sigma_1^{\,2} - \sigma_2\sigma_1 + \sigma_2^{\,2}) = S_y^{\,2}$$

$$3\tau^2 = S_y^{\,2} \qquad S_{ys} = S_y / \sqrt{3} \rightarrow \boxed{S_{ys} = .577\, S_y}$$

Relationship between yield strength in tension and shear

If $\sigma_y = 0$, then $\sigma_1, \sigma_2 = \sigma_x/2 \pm \sqrt{[(\sigma_x)/2]^2 + (\tau_{xy})^2}$

the design equation can be written in terms of the dominant component stresses (due to bending and torsion)

$$\boxed{\left[(\sigma_x)^2 + 3(\tau_{xy})^2\right]^{1/2} = \frac{S_y}{n}}$$

Design Process

Distortion energy theory

$$\sigma' = \frac{S_y}{n}$$

Maximum shear stress theory

$$\tau_{max} = \frac{S_y}{2n}$$

- Select material: consider environment, density, availability $\rightarrow$ S_y, S_u
- Choose a safety factor $n \uparrow$ Size $\uparrow$ Weight $\uparrow$ Cost $\uparrow$

The selection of an appropriate safety factor should be based on the following:

- Degree of uncertainty about loading (type, magnitude and direction)
- Degree of uncertainty about material strength
- Uncertainties related to stress analysis
- Consequence of failure; human safety and economics
- Type of manufacturing process
- Codes and standards

Design Process

- Use n = 1.2 to 1.5 for reliable materials subjected to loads that can be determined with certainty.
- Use n = 1.5 to 2.5 for average materials subjected to loads that can be determined. Also, human safety and economics are not an issue.
- Use n = 3.0 to 4.0 for well known materials subjected to uncertain loads.

Design Process

- Select material, consider environment, density, availability $\rightarrow$ S_y , S_u
- Choose a safety factor
- Formulate the von Mises or maximum shear stress in terms of size.
- Use appropriate failure theory to calculate the size.

$$\sigma' = \frac{S_y}{n} \qquad \tau_{\max} = \frac{S_y}{2n}$$

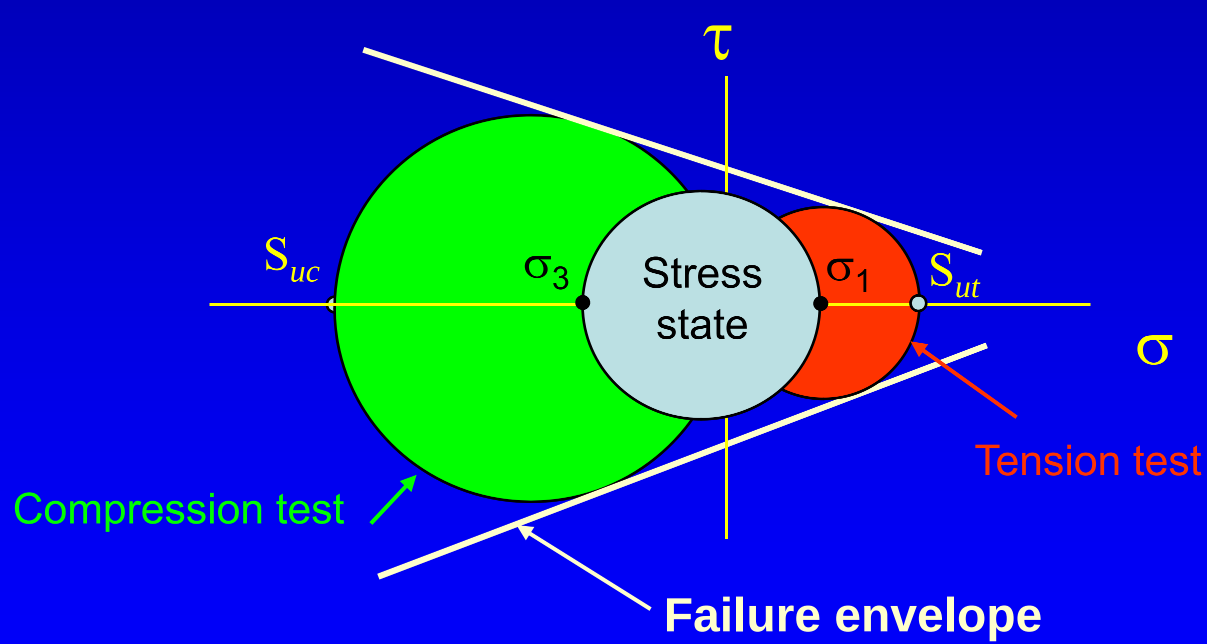
- Optimize for weight, size, or cost.

Failure Theories – Brittle Materials

One of the characteristics of a brittle material is that the ultimate strength in compression is much larger than ultimate strength in tension.

$$S_{uc} >> S_{ut}$$

Mohr's circles for compression and tension tests.

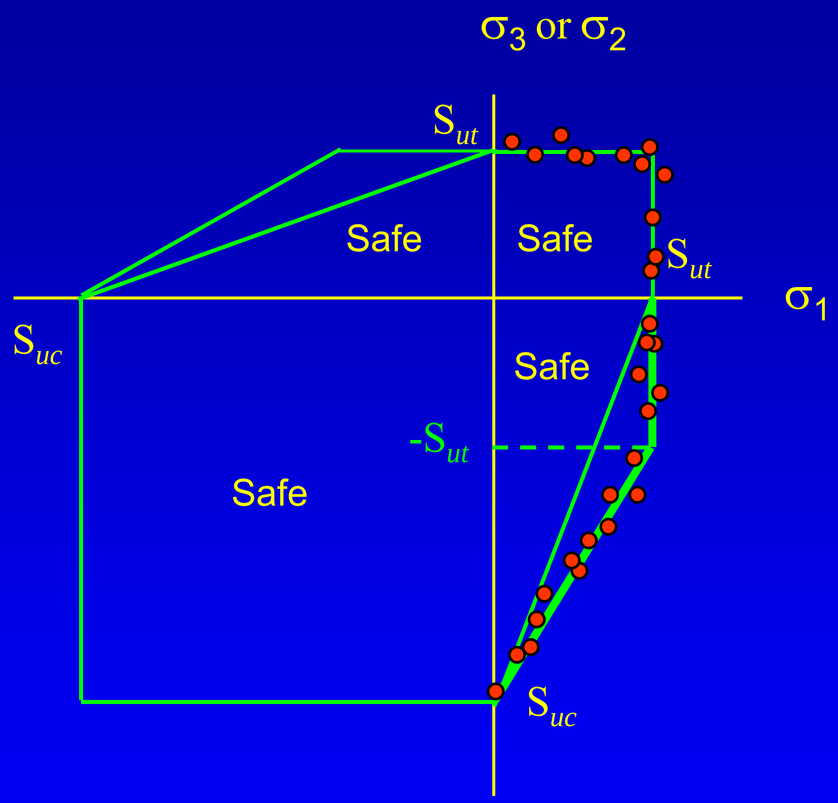


The component is safe if the state of stress falls inside the failure envelope.

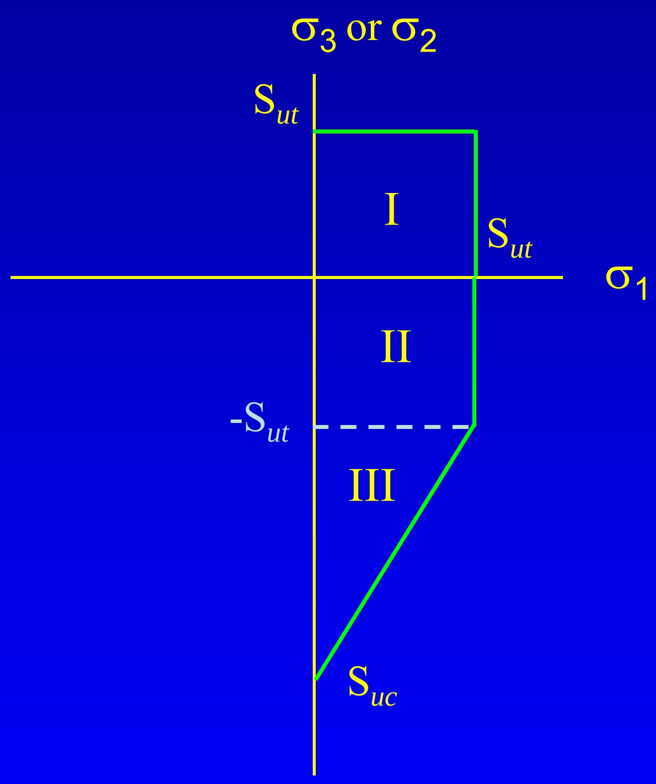
$$\sigma_1 > \sigma_3 \text{ and } \sigma_2 = 0$$

Failure Theories – Brittle Materials

Modified Coulomb-Mohr theory

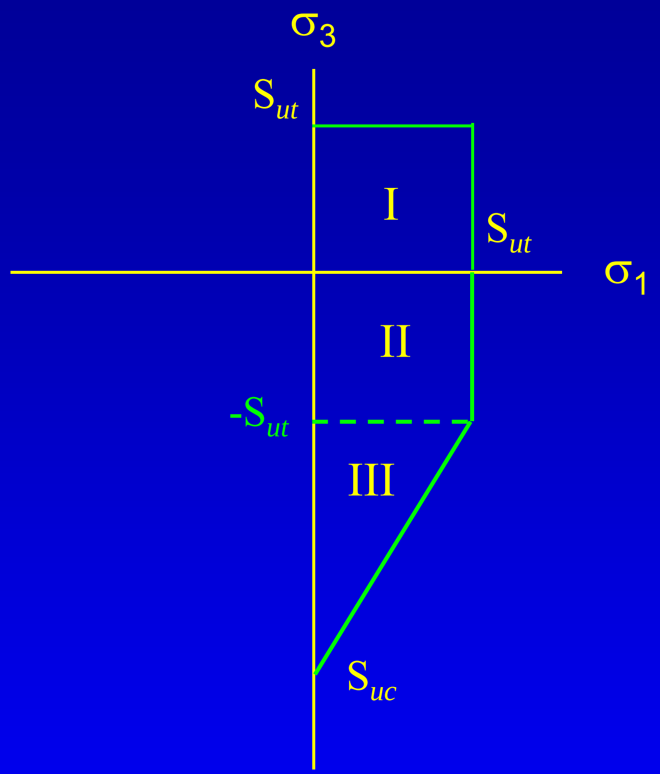


Cast iron data



Three design zones

Failure Theories – Brittle Materials



Zone I

$\sigma_1 > 0\,,\quad \sigma_2 > 0 \quad \text{and} \quad \sigma_1 > \sigma_2$

$$\sigma_1 = \frac{S_{ut}}{n}$$

Design equation

Zone II

$\sigma_1 > 0\,,\quad \sigma_2 < 0 \quad \text{and} \quad |\sigma_2| < S_{ut}$

$$\sigma_1 = \frac{S_{ut}}{n}$$

Design equation

Zone III

$\sigma_1 > 0\,,\quad \sigma_2 < 0 \quad \text{and} \quad |\sigma_2| > S_{ut}$

$$\sigma_1\left(\frac{1}{S_{ut}} - \frac{1}{S_{uc}}\right) - \frac{\sigma_2}{S_{uc}} = \frac{1}{n}$$

Design equation